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ON THE MOTION OF A PARTICLE OVER A ROUGH ROTATING PLANE*

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The motion of a particle over a rough horizontal plane that rotates with constant angular velocity about a vertical axis is studied. The particle is acted on by dry friction with the plane that satisfies Coulomb's law. The equations of motion are solved asymptotically close to the point where the particle stops. Various types of motion with stopping are found. The motions are simulated numerically over the full range of variation of the parameters and initial conditions. As a result, typical trajectories of particle motion are constructed. The integral manifold which isolates in phase space the motions with stopping and the motions in which the particle moves away to infinity is found. The phase trajectories are constructed in this manifold.

1. Formulation of the problem. We consider the motion of a particle P of mass m over a horizontal plane which moves with constant angular velocity ω about a vertical axis. We assume that the particle and plane interact according to Coulomb's of dry friction with coefficient of friction f. We connect a moving Cartesian coordinate system $Oxyz$ with the rotating plane, taking the vertical axis of rotation as the z axis. We assume for clarity that the plane rotates counterclockwise, so that $\omega > 0$.

We write the equations of planar motion of the particle P in the $Oxyz$ coordinate system taking into account the forces of inertia and friction as

$$x^{\cdot\cdot} = -fgv^{-1}x^{\cdot} + \omega^2 x + 2\omega y^{\cdot}, \quad y^{\cdot\cdot} = -fgv^{-1}y^{\cdot} + \omega^2 y - 2\omega x^{\cdot}$$
$$v = [(x^{\cdot})^2 + (y^{\cdot})^2]^{1/2} > 0 \tag{1.1}$$

Here, v is the velocity of P relative to the moving plane, g is the acceleration due to gravity, and the dots denote derivatives with respect to time.

We introduce polar coordinates r, φ in the xy plane, and denote by θ the angle between the relative velocity vector of P and the x axis (Fig.1). We then have

$$x = r\cos\varphi, \quad y = r\sin\varphi$$
$$x^{\cdot} = v\cos\theta, \quad y^{\cdot} = v\sin\theta$$

Transforming from variables $x, x^{\cdot}, y, y^{\cdot}$ in system (1.1) to variables r, φ, v, θ, we obtain the equations of motion

$$r^{\cdot} = v\cos(\theta - \varphi), \quad \varphi^{\cdot} = vr^{-1}\sin(\theta - \varphi)$$
$$v^{\cdot} = -fg + \omega^2 r\cos(\theta - \varphi), \quad \theta^{\cdot} = -2\omega - \omega^2 rv^{-1}\sin(\theta - \varphi) \tag{1.2}$$

If the particle is at rest relative to the plane $(v = 0)$ at some instant, and we have the condition

$$r \leqslant a = fg\omega^{-2} \tag{1.3}$$

then the particle will remain at rest, since the centrifugal force $m\omega^2 r$ does not in this case exceed the rest friction force fmg.

We will change to dimensionless variables, taking as unit length the radius a of the rest zone of (1.3), and as the unit of time ω^{-1}. In (1.2), we put

$$r = ar', \quad t = \omega^{-1}t', \quad v = a\omega^{-1}v' \tag{1.4}$$

Further, instead of θ we take the angle α between the velocity and radius vectors (Fig.1)

$$\alpha = \theta - \varphi \tag{1.5}$$

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After reduction of Eqs.(1.4) and (1.5) using (1.3), Eqs.(1.2) become (we omit the prime on the dimensionless variables; the dot denotes the derivative with respect to dimensionless time t')

$$r^{\cdot} = v\cos\alpha, \quad v^{\cdot} = r\cos\alpha - 1$$
$$\alpha^{\cdot} = -2 - (rv^{-1} + vr^{-1})\sin\alpha, \qquad \varphi^{\cdot} = vr^{-1}\sin\alpha\,(v \geqslant 0,\ r \geqslant 0) \tag{1.6}$$

while the initial conditions are

$$t = t_0, \quad r = r_0, \quad v = v_0, \quad \alpha = \alpha_0, \quad \varphi = \varphi_0 \tag{1.7}$$

Note that the first three equations of (1.6) do not contain φ, and can therefore be integrated independently of the fourth equation. After integration, the cyclical coordinate φ is found by a quadrature.

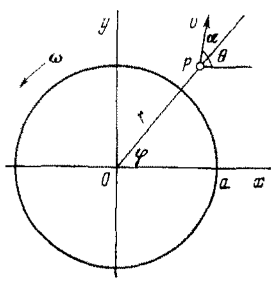


Fig.1

Since there are no parameters in (1.6), its solution depends only on the initial Conditions (1.7). The radius of the rest zone for system (1.6) is equal to unity, i.e., if $r_0 \leqslant 1$, $v_0 = 0$ in (1.7), then $r \equiv r_0$, $v \equiv 0$, $\alpha \equiv \alpha_0$, $\varphi \equiv \varphi_0$ for all $t \geqslant t_0$.

As $v \to 0$, Eqs.(1.6) have singularities. We shall analyse these in Sect.2 by finding the asymptotic form of the motion close to the instants of stopping, i.e., when v is close to zero. Notice that the singularities of (1.6) are unimportant as $r \to 0$, and are connected with the change to polar coordinates; they are not present for Eqs.(1.1) in Cartesian coordinates.

In Sect.3 we give data of the numerical integration of system (1.6) for different typical initial Conditions (1.7). In Sect.4 we construct the integral manifold which separates the motions with stopping, in which $v \equiv 0$ for sufficiently large $t > t^*$, from the motions which depart to infinity, in which $r \to \infty$, $v \to \infty$ as $t \to \infty$.

2. *The asymptotic behaviour of motions with stopping.* Let the motion stop at some instant $t = t^*$, $v(t^*) = 0$. We put

$$r(t^*) = r^*,\ v(t^*) = 0,\ \alpha(t^*) = \alpha^*,\ \varphi(t^*) = \varphi^*,\ t - t^* = \tau \tag{2.1}$$

We shall consider both the stopping (braking) process, for which $\tau \leqslant 0$, in (2.1), and the start of motion, for which $\tau \geqslant 0$. If $r^* \leqslant 1$, only braking is possible; here, for $\tau \geqslant 0$, we have $r \equiv r^*$, $v \equiv 0$, $\alpha \equiv \alpha^*$, $\varphi \equiv \varphi^*$. If $r^* > 1$, the motion must necessarily continue after stopping at the instant t^*, i.e. $v > 0$ for $\tau > 0$.

Using (2.1), we obtain from the second equation of (1.6):

$$v = (r^*\cos\alpha^* - 1)\tau + \ldots \tag{2.2}$$

where throughout the dots denote small higher-order terms in τ. On substituting (2.1) and (2.2) into the third equation of (1.6), we have

$$\alpha^{\cdot} = -2 - r^*(r^*\cos\alpha^* - 1)^{-1}\sin\alpha^*\tau^{-1} + \ldots$$

Hence, if $r^*\sin\alpha^* \neq 0$, we have $\alpha^{\cdot} = O(\tau^{-1})$, $\alpha = O(\ln|\tau|)$, which contradicts the condition $\alpha \to \alpha^*$ as $\tau \to 0$ of (2.1). Thus the stopping Condition (2.1) can only hold if either $r^* = 0$, or else $\sin\alpha^* = 0$. We can assume without loss of generality that $0 \leqslant \alpha < 2\pi$. Stopping can then occur in the cases $\alpha^* = 0$, $\alpha^* = \pi$, and $r^* = 0$.

We substitute expansion (2.2) in the first of Eqs.(1.6) and integrate it for small τ with the initial condition $r(t^*) = r^*$ of (2.1). In the case $\alpha^* = 0$ we obtain

$$r = r^* + {}^1/_2(r^* - 1)\tau^2 + \ldots, \quad v = (r^* - 1)\tau + \ldots \tag{2.3}$$

In the case $\alpha^* = \pi$ we have

$$r = r^* + {}^1/_2(r^* + 1)\tau^2 + \ldots, \quad v = -(r^* + 1)\tau + \ldots \tag{2.4}$$

We substitute expansion (2.3) for $\alpha^* = 0$ into the third of Eqs.(1.6). For small τ, noting that α is small, we obtain, apart from higher-order terms, for $r^* > 0$, $r^* \neq 1$,

$$\alpha^{\cdot} = -2 - r^*(r^* - 1)^{-1}\tau^{-1}\alpha + \ldots \tag{2.5}$$

similarly, we substitute expansion (2.4) for $\alpha^* = \pi$ into the third of Eqs.(1.6). In this equation we put

$$\alpha = \pi + \beta \tag{2.6}$$

and simplify it by noting that τ and β are small and assuming that $r^* > 0$

$$\beta^{\cdot} = -2 - r^*(r^* + 1)^{-1}\tau^{-1}\beta + \ldots \tag{2.7}$$

We note that the case $r^* = 0$ refers to stopping at the origin. The velocity vector with $\tau \leqslant 0$ must then be in the opposite direction to the radius vectors, i.e., $\alpha^* = \pi$ for

$r^* = 0$. In the case $r^* = 0$, we can write (2.4) as

$$r = {}^1/_2\tau^2 + \ldots, \quad v = -\tau + \ldots \tag{2.8}$$

while the third equation of (1.6) instead of (2.7) takes the form

$$\beta^{\cdot} = -2 - 2\tau^{-1}\beta + \ldots \tag{2.9}$$

Eqs.(2.5), (2.7) and (2.9), obtained for the cases $\alpha^* = 0, \alpha^* = \pi, r^* = 0$, are of the same type:

$$q^{\cdot} = -2 + \gamma q\tau^{-1} \tag{2.10}$$

where q is a variable, and γ is a constant.

We write the solution of the linear Eq.(2.10)

$$q = C\,|\,\tau\,|^{\gamma} + 2\,(\gamma - 1)^{-1}\tau \quad (\gamma \neq 1) \tag{2.11}$$

Here and below, C is an arbitrary constant of integration. Depending on γ, either the first or the second term in (2.11) may be the main term for q as $\tau \to 0$. We therefore obtain separately below the asymptotic behaviours for the different cases. We use relations (2.3)-(2.9) for r, v, α, and find the angle φ by a quarature with the aid of the last of Eqs.(1.6) and the relevant initial Condition (2.1).

1) $$0 < r^* < {}^1/_2, \quad \alpha^* = 0$$

Comparing (2.5) and (2.10), we obtain

$$\gamma = r^*\,(1 - r^*)^{-1} \tag{2.12}$$

Since $0 < \gamma < 1$ in this case, the first term in the solution (2.11) is the principal term. From (2.3), the condition $v \geqslant 0$ is only satisfied here for $\tau \leqslant 0$. We have (2.3) for r and v, while α and φ are given by

$$\begin{gathered} \alpha = C\,(-\tau)^{\gamma} + \ldots \\ \varphi = \varphi^* - (1 - r^*)^2(r^*)^{-1}(2 - r^*)^{-1}C\,(-\tau)^{\gamma+2} + \ldots \\ 0 < r^* < {}^1/_2, \quad 0 < \gamma < 1, \quad \tau \leqslant 0 \end{gathered} \tag{2.13}$$

2) $$r^* = {}^1/_2, \quad \alpha^* = 0$$

Integration of the equation with $\gamma = 1$ gives, instead of (2.11), $\alpha = C\tau - 2\tau \ln|\,\tau\,|$. Retaining the principal terms of the expansions, we obtain (omitting the terms with the constant C)

$$\begin{gathered} r = {}^1/_2 - {}^1/_4\tau^2 + \ldots, \quad v = -{}^1/_2\tau + \ldots, \quad \alpha = -2\tau \ln|\,\tau\,| + \ldots, \\ \varphi = \varphi^* + {}^2/_3\tau^3 \ln|\,\tau\,| + \ldots \\ r^* = {}^1/_2, \quad \tau \leqslant 0 \end{gathered} \tag{2.14}$$

3) $${}^1/_2 < r^* < 1, \quad \alpha^* = 0$$

In accordance with (2.12), we have $\gamma > 1$ here, so that we retain the second term in (2.11) (we omit the term with constant C). Hence

$$\begin{gathered} \alpha = 2\,(1 - r^*)(2r^* - 1)^{-1}\tau + \ldots \\ \varphi = \varphi^* - {}^2/_3\,(1 - r^*)^2(r^*)^{-1}\,(2r^* - 1)^{-1}\tau^3 + \ldots \\ {}^1/_2 < r^* < 1, \quad \tau \leqslant 0 \end{gathered} \tag{2.15}$$

For r and v we have (2.3).

4) $$r^* > 1, \quad \alpha^* = 0$$

By (2.12), we have $\gamma < 0$, so that, with $C \neq 0$, the first term in (2.11) tends to infinity as $\tau \to 0$. But this contradicts the condition $\alpha \to 0$ as $\tau \to 0$, which must hold here $(\alpha^* = 0)$. Thus the solution that satisfies this condition will hold only for $C = 0$. While the solution is given by the same expressions (2.3), (2.15) as in the previous case, we can see here from (2.3) that the condition $v \geqslant 0$ only holds when $\tau \geqslant 0$. Thus, whereas cases 1)-3) refer to braking of the motion, case 4) refers to starting up from a state of rest. Moreover, in cases 1)-3) the asymptotic behaviour is not unique (the constant C is arbitrary), whereas in case 4) it is unique $(C = 0)$.

5) $$r^* > 0, \quad \alpha^* = \pi$$

Eq.(2.7) for β has the form (2.10), where

$$\gamma = -r^* (r^* + 1)^{-1} < 0 \tag{2.16}$$

Hence, in the same way as in case 4), the solution (2.11) tends to zero as $\tau \to 0$ only if $C = 0$. The variables r, v are given by (2.4), where $v > 0$ only when $\tau \leqslant 0$. For α, φ, we obtain by (2.6), (2.11), (2.16)

$$\alpha = \pi - 2(r^* + 1)(2r^* + 1)^{-1}\tau + \ldots \tag{2.17}$$
$$\varphi = \varphi^* - {}^2/_3 (r^* + 1)^2 (r^*)^{-1} (2r^* + 1)^{-1}\tau^3 + \ldots; \quad \tau \leqslant 0$$

6) $$r^* = 0, \ \alpha^* = \pi$$

Eq.(2.9) for β has the form (2.10), where $\gamma = -2$. As in cases 4), 5), we have to put $C = 0$ in solution (2.11). The variables r, v are given by (2.8), where $\tau \leqslant 0$, while

$$\alpha = \pi - {}^2/_3\tau + \ldots, \quad \varphi = \varphi^* - {}^4/_3\tau + \ldots; \quad \tau \leqslant 0 \tag{2.18}$$

Here, as in cases 4) and 5), the solution is unique,

7) $$r^* = 1, \ \alpha^* = 0$$

This case is singular: an infinitly large term appears in (2.5). Thus the assumption that stopping occurs at a finite instant t^* leads to a contradiction. This suggests that we should look for a solution in which the stopping conditions are reached asymptotically, i.e.,

$$r \to 1, \ v \to 0, \ \alpha \to 0 \text{ as } t \to \infty \tag{2.19}$$

On replacing $\cos\alpha$ by 1 in the first two of Eqs.(1.6), we obtain the simplified system

$$r^{\cdot} = v, \quad v^{\cdot} = r - 1$$

Its solution under Conditions (2.19) is

$$r = 1 - be^{-t}, \quad v = be^{-t}, \quad b > 0 \tag{2.20}$$

where b is a constant. Substituting (2.20) into the third of Eqs.(1.6), we obtain, apart from higher-order terms

$$\alpha^{\cdot} + 2 + b^{-1}e^t\alpha + \ldots = 0$$

The principal (second and third) terms of this equation cancel each other if we take

$$\alpha = -2be^{-t} + \ldots \tag{2.21}$$

It can now be seen, using (2.21), that the replacement of $\cos\alpha$ by 1 in the first two of Eqs.(1.6), has introduced an error into the higher-order terms.

Relations (2.20) and (2.21) give the principal terms of the asymptotic expansions. Notice that these expansions can be obtained in a regular way, by seeking r, v, α formally as power series in the argument e^{-t}, but we shall not dwell on this here. We shall also quote the principal terms of the expansion for the angle φ, obtained from the last of Eqs.(1.6) and from (2.20), (2.21),

$$\varphi = \varphi^* + b^2e^{-2t} + \ldots \tag{2.22}$$

The above cases 1)-7) exhaust all the possible motions with stopping, i.e., the motions with which the velocity v vanishes at a finite instant t^* or as $t \to \infty$. Only the principal terms of the expansions have been found, though the subsequent terms can be obtained if required. The data on the asymptotic solutions are collected in Table 1, where we give for each case the values of τ for which this case holds (in case 7) $t \to \infty$); the range of values of r^*; the limiting values of α; the signs of $r - r^*$; the signs of $\varphi - \varphi^*$; the number N of solutions of the given type for fixed α^*, r^*, φ^*; the number of the expression that gives the solution in this case; the number of the figure that illustrates the case and is described in Sect.3. Notice that the signs of $r - r^*, \alpha - \alpha^*, \varphi - \varphi^*$, are always, except for $\alpha - \alpha^*$, $\varphi - \varphi^*$, in case 1), given uniquely by the expressions. In case 1) there is arbitrariness due to the arbitrariness of the constant C.

Table 1

No.	τ	r^*	α	$r - r^*$	$\varphi - \varphi^*$	N	Expression	Fig.
1	$-$	$(0, {}^1/_2)$	± 0	$-$	$\pm$	∞	(2.3), (2.13)	4, a, b
2	$-$	${}^1/_2$	-0	$-$	$+$	∞	(2.14)	4, c
3	$-$	$({}^1/_2, 1)$	-0	$-$	$+$	∞	(2.3), (2.15)	4, d
4	$+$	$(1, \infty)$	-0	$+$	$-$	1	(2.3), (2.15)	6, a
5	$-$	$(0, \infty)$	$\pi + 0$	$+$	$+$	1	(2.4), (2.17)	5, a 6, a
6	$-$	0	$\pi + 0$	$+$	$+$	1	(2.8), (2.18)	3, b
7	$-\infty$	1	-0	$-$	$+$	∞	(2.20)—(2.22)	3, a, c

The form of our solutions is shown in Fig.2, where, to fix our ideas, we take $\varphi^* = 0$.

Numerical simulation to high accuracy confirmed the validity of our asymptotic expansions. Note that these expansions can be used when integrating the equations numerically close to an instant of stopping.

3. ***Numerical simulation.*** For the numerical integration of system (1.6) under the initial Conditions (1.7), we made up a package of subroutines in $PL/1$ (0), language, whereby the motions could be modelled over a wide range of initial data in the intervals $r_0 \in [0, \infty)$, $v_0 \in [0, \infty)$, $\alpha_0 \in [0, 2\pi)$. The numerical integration was performed both in forward and reverse time with a variable time step, in accordance with the Runge-Kutta scheme. There was no loss of generality in taking $t_0 = 0$, $\varphi_0 = 0$. Special attention was paid to the trajectories with stopping. The results of integration were obtained as curves of $r(t)$, $v(t)$, $\alpha(t)$, $\varphi(t)$, and as trajectories of the particle motion in a rotating and fixed Cartesian coordinate system.

For instance, in Figs.3-7 below, the continuous lines are the trajectories of the particle P, in the fixed coordinate system, and the broken lines are the trajectories in the rotating system. For clarity, the dot-dash circle shows the boundary of the rest zone $r \leqslant 1$. In Fig.3 we show motions with stops, the trajectories of which begin and end either on the boundary of the rest zone or at the origin of coordinates.

In Fig.3,a we show the trajectory that starts at the origin and ends on the unit circle (the boundary of the rest zone). The corresponding initial data are

$$r_0 = 0,\ v_0 = v_{01} = 1.165,\ \alpha_0 = 0 \tag{3.1}$$

In (3.1), v_{01} was chosen so that the particle P stops as $r \to 1$; as $t \to \infty$, we have the asymptotic behaviour for case 7), see (2.20)-(2.22). If $v_0 > v_{01}$, $r_0 = 0$ the particle departs to infinity, while if $v_0 < v_{01}$, it stops inside the circle $r \leqslant 1$.

In Fig.3,b we show the trajectory which starts on the unit circle and ends at the origin. The corresponding initial data, found by integration in reverse time, are

$$r_0 = 1,\ v_0 = v_{02} = 1.845,\ \alpha_0 = -135.80^\circ \tag{3.2}$$

Stopping occurs at the instant $t^* = 1.4$; as $t \to t^*$ we have the asymptotic behaviour for case 6), see (2.18).

In Fig.3,c we show the trajectory that starts on the unit circle, passes through the origin with non-zero velocity v_{01}, and ends on the unit circle according to the asymptotic behaviour for case 7). The initial data, found by integration in reverse time, are $r_0 = 1$; $v_0 = 2.12$; $\alpha_0 = -148.08^\circ$.

In Fig.4 we show the motions with stopping inside the rest zone. In Fig.4,a,b we see the trajectories with stopping for $r^* \in (0, 1/2)$. For Fig.4,a we have $\alpha(t^* - 0) < 0$, and for Fig. 4,b, $\alpha(t^* - 0) > 0$ in accordance with the asymptotic behaviour for case 1). The initial data for Fig.4,a, found by integration in reverse time, are: $r_0 = 1.2$; $v_0 = 2.21$; $\alpha_0 = -140.11^\circ$, and for Fig. 4,b: $r_0 = 1.2$; $v_0 = 1.89$; $\alpha_0 = -147^\circ$. In Fig.4,c we see the motion with stopping for $r^* = 1/2$ in accordance with the asymptotic behaviour for Case 2). The initial data are here: $r_0 = 1.2$; $v_0 = 2.40$; $\alpha_0 = -131.17^\circ$. In Fig.4,d we show the trajectory with stopping for $r^* \in (1/2, 1)$ (the asymptotic behaviour for Case 3)), with the initial data $r_0 = 1.2$; $v_0 = 2.52$; $\alpha_0 = -125^\circ$.

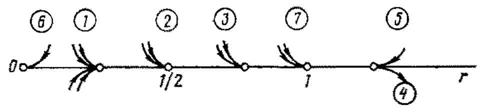
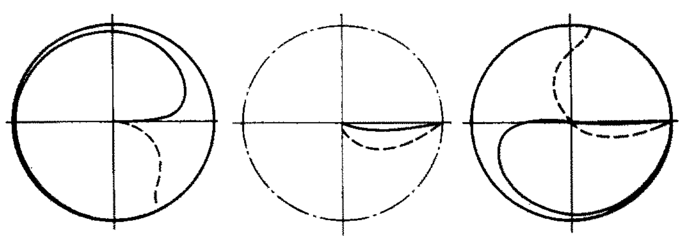


Fig.2



a b c

Fig.3

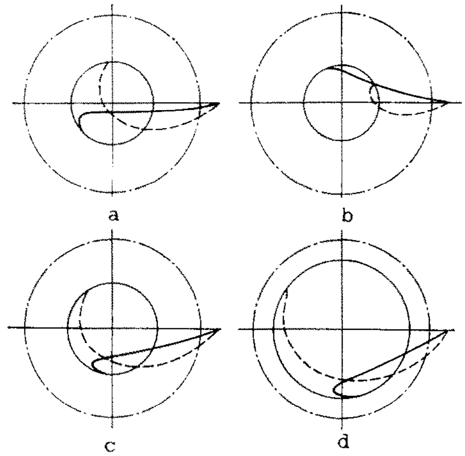


Fig.4

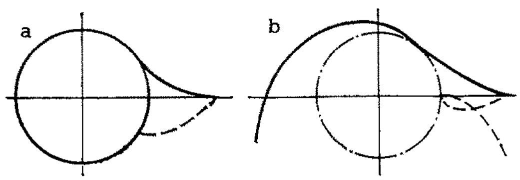


Fig.5

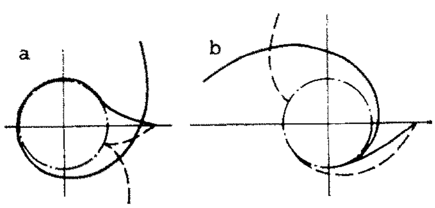


Fig.6

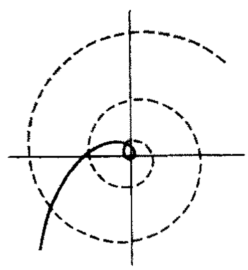


Fig.7

In Fig.5 we show the trajectory that approaches the rest zone from outside. In Fig.5,a we see the motion with stopping on the unit circle. As distinct from Fig.3,a,c, the stopping occurs here outside the rest zone in accordance with the asymptotic behaviour for case 5). The corresponding initial data are $r_0 = 2$; $v_0 = 2.38$; $\alpha_0 = -127°$. It must be said that this motion (with stopping for $\alpha^* = \pi + 0$) is unstable, and just a slight change of the above initial data leads to the particle not reaching the rest zone and departing to infinity, as shown in Fig.5,b, obtained for $r_0 = 2$; $v_0 = 2.3$; $\alpha_0 = -145°$.

In Fig.6 we see the motion with stopping outside the rest zone, after which particle P continues to move and departs to infinity. In Fig.6,a we have the case when stopping occurs in the rotating coordinate system ($v = 0$ at the instant t^*), and in Fig.6,b, the case when the absolute velocity of P vanishes (here $v \neq 0$). The initial data for Fig.6,a are $r_0 = 2$; $v_0 = 2.4$; $\alpha_0 = -125°$, and for Fig.6,b: $r_0 = 2$; $v_0 = 3.5$; $\alpha_0 = -115°$.

In Fig.7 we see the trajectory for which the radius and velocity tend to infinity as $t \to \infty$. the initial conditions for Fig.7 are $r_0 = 1$, $v_0 = 1$, $\alpha_0 = 90°$. The exact solution, which, departs to infinity, was obtained in /1/; the trajectory in the fixed coordinate system is here a logarithmic spiral; it was also shown that this solution, is limiting for all motions which depart to infinity. In a moving coordinate system, the asymptotic form of the motions which depart to infinity is

$$r = (3\sqrt{2})^{-1}t^2 + \ldots, \quad v = (3\sqrt{2})^{-1}\,t^2 + \ldots, \qquad \alpha = -\pi/2 + 2t^{-1} + \ldots, \qquad (3.3)$$
$$\varphi = C - t + \ldots$$

Our numerical solution is in accord with this asymptotic form.

4. ***The separating integral manifold.*** For any initial conditions, the motion of P is one of two types: either the particle departs to infinity in accordance with asymptotic form (3.3), or the velocity of P tends to zero, and it stops in the circle $r \leqslant 1$. In the latter case we have one of the asymptotic behaviours corresponding to cases 1)-3), 5)-7).

The "boundary" between these types of motion are those which end on the circle $r = 1$. It can be seen from Table 1 that these boundary motions are type 5) or 7). In case 5), the particle approaches the circle $r = 1$ from outside in finite time, and there is just one such trajectory for each point of the circle. In case 7), P approaches the circle $r = 1$ from inside in infinite time; and an infinite number of trajectories enter each point of the circle.

Consider the motion of P in three-dimensional phase space u, w, r, ignoring the cyclical variable φ and putting

$$u = v\cos\alpha, \quad w = v\sin\alpha \qquad (4.1)$$

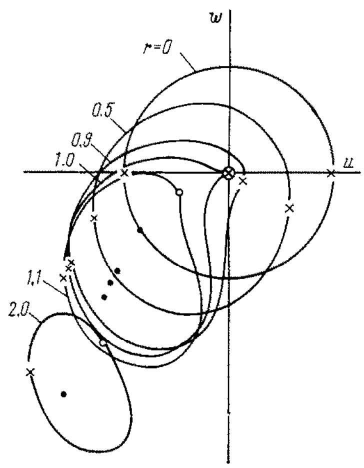


Fig.8

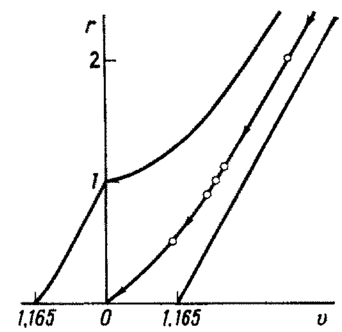


Fig.9

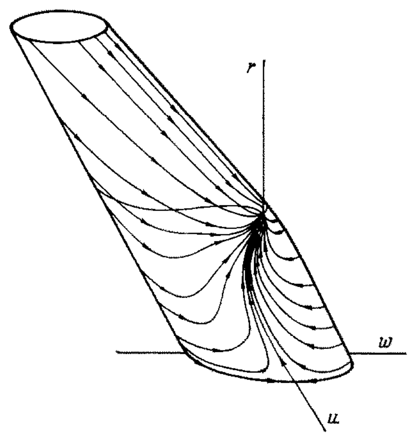


Fig.10

The variables (4.1) are respectively equal to the projections of the relative velocity of P onto the radius vector and the perpendicular to it. The domain of possible motions in the variables u, w, r is the half-space $r \geqslant 0$.

We fix r and find numerically the pairs of u, w, such that the motions starting with these u, w, r, end on the circle $r = 1$. We thus obtain a section by the plane $r = \text{const}$ of the surface S, which separates the motions that depart to infinity from these which stop in the circle $r \leqslant 1$. We found the section of S by many calculations; these sections are shown in Fig.8. The numbers on the curves show the relevant values of $r \geqslant 0$. The domains inside the curves correspond to points which lie on trajectories with stopping, and the exterior domains, to unlimited motions.

We mark out specially on the sections the points which lie on the trajectories shown in Fig.3,b,c and Fig.5,a. The points which lie on the trajectories of Fig.3,c (with stopping on the boundary of the rest zone and with passage through the origin), are marked by a cross. The points of the trajectories of Fig.5,a (stopping outside the rest zone) are marked by a light circle, and the points of Fig.3,b by a dark circle. In Fig.8 the points marked by a dark circle can be joined by a curve l which passes through the origin. The curve l is the projection of the trajectory of Fig.3,b onto the uw plane.

In Fig.9 we mark by heavy lines the section of the surface S by the cylindrical surface with director l and generators parallel to the r axis.

The section of S by the plane $r = 0$ is obviously a circle of radius v_{01}, see (3.1). As r increases, the section moves, deforms and decreases. Notice that, for $r \leqslant 1$, the sections contain the point $u = w = 0$, since, with the initial data $v = 0, r \leqslant 1$ the particle P remains at rest. With $r > 1$, the sections no longer contain the origin of the uw plane.

The entire surface S consists of phase trajectories: if the initial point lies on S, then all the trajectories lie on S. The calculated picture of these phase trajectories is shown in Fig.10 in space (u, w, r). The arrows indicate the direction is which the time varies.

The surface S itself, i.e., the set of its sections, can be obtained as follows. First, by selecting v_0 and α_0 with $r = 1.2$, in the dialogue mode, we constructed the section K_0 by graphical interpolation and smoothing of more than 100 points. Then, from 50 points of the curve, by numerical integration, we obtain the trajectories in forward and reverse time. Using the former, we can construct the sections (with step $\Delta r = 0.1$) with $r \leqslant 1.2$ and thus obtain the lower part of the surface S. The trajectories which issue from points of section K_0 in reverse time can be used to obtain the section with $r > 1.2$ and we thus complete the formation of the integral manifold. The trajectories themselves which form S are then drawn.

It can be seen from Fig.10 that all the trajectories which start on S will end at a singular point with coordinates $u = w = 0, r = 1$, that corresponds to stopping on the boundary of the rest zone. Only one trajectory enters the singular point from above. It corresponds to the asymptotic behaviour of case 5), i.e., stopping outside the rest zone. All the other trajectories intersect the section $r = 1$ and approach the singular point from below. The trajectories which lie on one side of the broken line correspond to entry into the rest zone with $0 < \alpha < \pi$, and on the other side, with $\pi < \alpha < 2\pi$.

There is a further singular trajectory on the surface S, i.e., the separatrix corresponding to the motion of Fig.3,c. It hits the section $r = 0$ then instantaneously (for $\Delta t = 0$) jumps from the point (-1.165, 0, 0) to the point (1.165, 0, 0), which corresponds to passage of P through the origin, for which $|\Delta\alpha| = \pi$. The separatrix, see Fig.10, then enters the singular point (0, 0, 1). All the other trajectories are divided by the separatrix into two groups.

To sum up, we have described all the trajectories of motions with stopping, which form an integral manifold.

REFERENCES

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